Non-commutative Rational Functions and Atiyah Property

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IWOTA, Shanghai
July 23, 2018

Supported by the ERC Advanced Grant "Non-commutative distributions in free probability"

European Research Council
Established by the European Commission

Strong Atiyah Conjecture

Definition

Let G be a discrete torsion free group and $L(G) \subseteq B(l^2(G))$ its left regular representation, then for any matrix $A \in M_N(\mathbb{C}[G])$, viewed as an operator in $M_N(L(G))$, we can define its *rank*

$$\operatorname{rank} A := (\operatorname{Tr}_N \otimes \tau) p_{\overline{\operatorname{im} A}} \in [0, N],$$

where τ is the trace on $L(G)$ given by the identity of G, Tr_N is the unormalized trace of $M_N(\mathbb{C})$, and $p_{\overline{\operatorname{im} A}}$ is the projection onto the closure of the image of A.

A torsion free group G satisfies the *strong Atiyah conjecture* if for any matrix $A \in M_N(\mathbb{C}G)$, its rank

$$\operatorname{rank} A \in \mathbb{Z}.$$

Strong Atiyah Conjecture

Remarks

1. In general, G is not necessary to be torsion free and then $\mathbb{Z}$ need to be modified according to some quantity of G.
2. This is one of the various formulations of the Atiyah conjecture, which arose in the work of Atiyah, *Elliptic operators and compact groups*, 1974. It asks whether some analytic L^2-Betti numbers are always rational numbers for certain Riemannian manifolds.
3. A big class of groups, included free groups $\mathbb{F}_n$, is known to satisfy the strong Atiyah conjecture.
4. Actually, we can generally consider a tuple of operators in a finite von Neumann algebra.

Strong Atiyah Property

Definition (D. Shlyakhtenko and P. Skoufranis, 2015)

Let $X = (X_1, \ldots, X_n)$ be a tuple of operators in a tracial W^*-probability space (M, τ) (namely, M is a finite von Neumann algebra with a faithful normal tracial state $\tau : M \to \mathbb{C}$), if for any matrix $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, the rank (defined as before) of its evaluation $A(X)$

$$\operatorname{rank} A(X) \in \mathbb{Z},$$

then we say X satisfies the strong Atiyah property.

Remarks

1. From the definition, $\operatorname{rank} A(X)$ can be any real number in $[0, N]$.
2. The generators of free group von Neumann algebras satisfy the strong Atiyah property (Linnell, 1993).
3. A tuple of non-atomic, freely independent operators satisfies the strong Atiyah property (Shlyakhtenko and Skoufranis, 2015).

Inner Rank

Definition

Let R be a unital ring. For any non-zero $A \in M_{m,n}(R)$, the *inner rank* of A is defined as the least positive integer r such that there are matrices $P \in M_{m,r}(A)$, $Q \in M_{r,n}(A)$ satisfying a factorization

$$A = PQ.$$

We denote this number by $\rho(A)$.
In particular, if $\rho(A) = \min\{m, n\}$, namely, if there is no such factorization with $r < \min\{m, n\}$, then A is called a *full* matrix over R.

Example:

$$A = \begin{pmatrix} y^2 & yxy \\ yxy & yx^2y \end{pmatrix} = \begin{pmatrix} y \\ yx \end{pmatrix} \begin{pmatrix} y & xy \end{pmatrix}$$

has inner rank $\rho(A) = 1$ over $\mathbb{C}\langle x, y \rangle$.

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Remarks

1. It is a natural generalization of the notion of rank to noncommutative ring.
2. If $R = \mathbb{C}$, then the inner rank is the matrix rank and a full matrix is non-singular matrix.

Rational Closure

Definition

Let $\varphi : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathcal{A}$ be a homomorphism into a unital algebra $\mathcal{A}$. The rational closure of $\mathbb{C}\langle x_1, \dots, x_n \rangle$ with respect to φ is the set of all elements given by

$$u\varphi(A)^{-1}v,$$

where $A \in M_k(\mathbb{C}\langle x_1, \dots, x_n \rangle)$ has its image invertible in $M_k(\mathcal{A})$ under matricial amplifications of φ and $u \in M_{1,k}(\mathbb{C})$, $v \in M_{k,1}(\mathbb{C})$ are scalar-valued row and column vectors respectively.

Remarks

1. A rational closure R is always a subalgebra of $\mathcal{A}$ which contains the image $\varphi(\mathbb{C}\langle x_1, \dots, x_n \rangle)$. Moreover, R is stable with respect to taking inverses, i.e., if $r \in R$ and r is invertible in $\mathcal{A}$, then $r^{-1} \in R$.
2. If φ is the evaluation map ev_X induced by a tuple of operators X, then the rational closure is the "evaluation of rational functions".

Rational Closure

Rational closure of a tuple of operators $X = (X_1, \ldots, X_n)$

Let $\mathcal{A}(M)$ be the $*$-algebra of closed and densely defined operators affiliated with M, and $\mathrm{ev}_X : \mathbb{C}\langle x_1, \ldots, x_n\rangle \to M \subseteq \mathcal{A}(M)$ the evaluation map at tuple X. Then the rational closure R with respect to $\mathrm{ev}_X : \mathbb{C}\langle x_1, \ldots, x_n\rangle \to \mathcal{A}(M)$ is a subalgebra of $\mathcal{A}(M)$ s.t. $r \in R$ is invertible in $\mathcal{A}(M) \implies r \in R$. Hence, for example, given $p_1, p_2 \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$,

$$(p_1(X) - p_2(X)^{-1})^{-1} \in \mathcal{A}(M) \implies R \ni (p_1(X) - p_2(X)^{-1})^{-1} \sim (p_1 - p_2^{-1})^{-1}(X).$$

Strong Atiyah Property

Theorem (T. Mai, R. Speicher, Y., 2018)

Let $X = (X_1, \ldots, X_n)$ be a tuple of operators in a tracial W^*-probability space (M, τ), and R is the rational closure of $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ with respect to evaluation map $\mathrm{ev}_X : \mathbb{C}\langle x_1, \ldots, x_n\rangle \to \mathcal{A}(M)$. Then the followings are equivalent:

1. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$: if $A(X)$ is full over R, then $A(X)$ is invertible as an unbounded operator in $M_N(\mathcal{A}(M))$.*
2. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$:*
$$\operatorname{rank} A(X) = \rho_R(A(X)).$$
3. *The rational closure R is a division ring, i.e., each element in R is invertible in $\mathcal{A}(M)$.*
4. *X satisfies the strong Atiyah property, i.e., for any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, we have $\operatorname{rank} A(X) \in \mathbb{N}$.*

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Remarks

1. Property (2) is a special case of Property (1) since $\operatorname{rank} A(X) = N$ iff $A(X)$ is invertible in $\mathcal{A}(M)$.
2. The equivalence of Property (3) and strong Atiyah conjecture is known for torsionfree groups (Linnell, 1992; Schick, 1999).
3. In Property (3), the quantity $\rho_R(A(X))$ on the right hand side is actually algebraic.

Strong Atiyah Property

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1. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$: if A is linear and full, then $A(X)$ is invertible as an unbounded operator in $M_N(\mathcal{A}(M))$.*
2. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$: if A is full, then $A(X)$ is invertible as an unbounded operator in $M_N(\mathcal{A}(M))$.*
3. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$:* $\operatorname{rank} A(X) = \rho(A)$.
4. *The set of all nc rational functions, $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, can be embedded into $\mathcal{A}(M)$ by the evaluation at tuple X.*

- In Property (4), $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, called *free field*, is the unique universal smallest division ring containing the ring of non-commutative polynomials (Amitsur 1966; Cohn 1971).

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3. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$:* $\operatorname{rank} A(X) = \rho(A)$.
4. *The set of all nc rational functions, $\mathbb{C}\left(\!\langle x_1, \dots, x_n\rangle\!\right)$, can be embedded into $\mathcal{A}(M)$ by the evaluation at tuple X.*

- In this case, the rational closure w.r.t $\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathcal{A}(M)$ is actually given by the evaluation of all rational functions at tuple X.

Strong Atiyah Property

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3. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$: $\operatorname{rank} A(X) = \rho(A)$.*
4. *The set of all nc rational functions, $\mathbb{C}\langle\!\langle x_1, \dots, x_n\rangle\!\rangle$, can be embedded into $\mathcal{A}(M)$ by the evaluation at tuple X.*

- In Property (3), $\rho(A)$ denotes the inner rank of A over $\mathbb{C}\langle x_1, \dots, x_n\rangle$, which only depends on the matrix A. As $\rho(A) \in \mathbb{N}$ by the definition, these equivalent properties are stronger than the strong Atiyah property and its equivalences in the previous theorem.

Strong Atiyah Property

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3. *For any $N \in \mathbb{N}$, $A \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$: $\operatorname{rank} A(X) = \rho(A)$.*
4. *The set of all nc rational functions, $\mathbb{C}\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle$, can be embedded into $\mathcal{A}(M)$ by the evaluation at tuple X.*

- Property (1) can be proved under some regular assumptions for tuple $X = (X_1, \ldots, X_n)$: finite free Fisher information $\Phi^*(X_1, \ldots, X_n) < \infty$ $\Longrightarrow$ maximal free entropy dimension $\delta^\star(X_1, \ldots, X_n) = n$

Absence of Zero Divisors/Atoms

Theorem (T. Mai, R. Speicher, Y., 2018)

- *Suppose that*

$$A = A_0 + A_1 x_1 + \cdots + A_n x_n \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle),\ A_i \in M(\mathbb{C})$$

  is a full matrix;

- *Suppose that $X = (X_1, \cdots, X_n)$ is a tuple of selfadjoint nc random variables in some tracial W^*-probability space (M, τ) s.t.*

$$\delta^\star(X_1, \ldots, X_n) = n;$$

then the evaluation $A(X)$ has zero divisors in $M_N(\mathrm{vN}(X))$.

Remarks

1. $A(X)$ is invertible in $M_N(\mathcal{A}(\mathrm{vN}(X)))$ since the projection $p_{\ker A(X)} = 0$.
2. If A is selfadjoint, then the distribution $\mu_{A(X)}$ has no atoms.

Let X, Y be freely independent semicircular random variables

Let

$$A = \begin{pmatrix} y^2 & yxy \\ yxy & yx^2y \end{pmatrix}$$

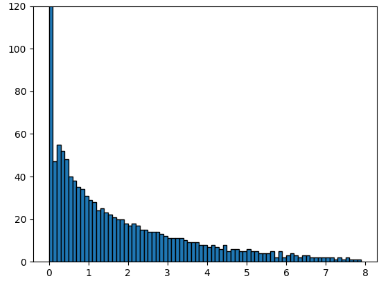
then

$$\operatorname{rank} A(X,Y) = \rho(A) = 1 < 2$$

$$\implies \mu_{A(X)}(\{0\}) = \frac{1}{2}$$

$$\implies \frac{\#\{\text{zero eigenvalues of } A_{(N)}\}}{N} \xrightarrow[N\to\infty]{} \frac{1}{2}$$

Let $X_{(N)}$, $Y_{(N)}$ be GUE random matrices with $N = 1000$



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then

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$$\Longrightarrow \mu_{A(X)}(\{0\}) = \frac{1}{2}$$

$$\Longrightarrow \frac{\#\{\text{zero eigenvalues of } A_{(N)}\}}{N}$$

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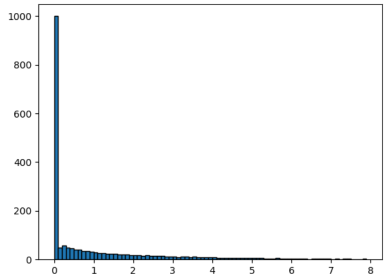

Let X, Y be freely independent semicircular random variables

Let

$$A = \begin{pmatrix} y^2 & yxy \\ yxy & yx^2y \end{pmatrix}, \ \lambda \in \mathbb{R}, \lambda \neq 0$$

then

$$\begin{aligned} & \operatorname{rank}(\lambda - A(X,Y)) \\ & = \rho(\lambda - A) = 2 \\ \implies & \mu_{A(X)}(\{\lambda\}) = 0 \\ \implies & \frac{\#\{\lambda \text{ eigenvalues of } A_{(N)}\}}{N} \\ & \xrightarrow[N\to\infty]{} 0 \end{aligned}$$

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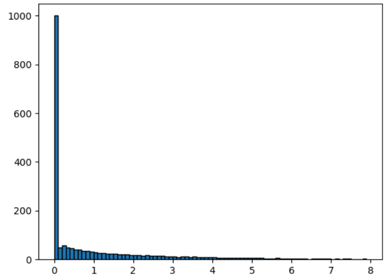


Rational functions, Atoms and Random Matrices

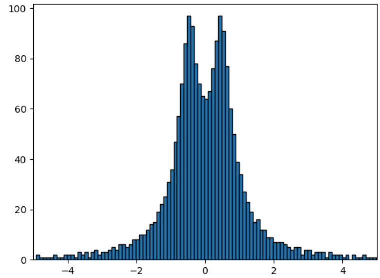
Let X, Y be freely independent semicircular random variables

Let

$$r = (x - y^{-1})^{-1} \neq 0$$

in $\mathbb{C}\langle x_1, \ldots, x_n \rangle$, then $\mu_{r(X,Y)}$ is non-vanishing and has no atoms.

Let $X_{(N)}$, $Y_{(N)}$ be GUE random matrices with $N = 1000$



Thank you!